Models for anisotropic spherical stellar systems with a central point mass and Keplerian-fall-off velocity dispersions

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ABSTRACT

We add to the lore of spherical, stellar-system models a two-parameter family with an anisotropic velocity dispersion, and a central point mass ("black hole"). The ratio of the tangential to radial dispersions is constant–and constitutes the first parameter–while each decreases with radius as $r^{-1/2}$. The second parameter is the ratio of the central point mass to the total mass. The Jeans equation is solved to give the density law in closed form: $\rho \propto (r/r_0)^{-\gamma}/[1+(r/r_0)^{3-\gamma}]^2$, where r_0 is an arbitrary scale factor. The two parameters enter the density law only through their combination γ. At the suggestion of Tremaine, we also explore models with only the root-sum-square of the velocities having a Keplerian run, but with a variable anisotropy ratio. This gives rise to a more versatile class of models, with analytic expressions for the density law and the dispersion runs, which contain more than one radius-scale parameter.

Subject headings: Celectial mechanics, stellar dynamics–star clusters, galaxies: ellipticals–galaxies: kinematics and dynamics–galaxies: structure

1. INTRODUCTION

Model spherical stellar systems are useful in various contexts, as models of elliptical galaxies, star clusters, cores of galaxies, etc., and also as test beds for various theoretical ideas. To the library of classic models such as the Emden polytropes, isothermal spheres, King, and Michie models [see e.g. Binney and Tremaine (1987) and references therein], were added, in recent years, the models of Jaffe (1983), of Bertin and Stiavelli (1989), and of Hernquist (1990) and their generalizations to "eta" models by Dehnen (1993), and by Tremaine et al. (1994).

There are various routes to constructing such spherical models: One may start from a distribution function (hereafter, DF) satisfying the Jeans theorem, and see if one gets a useful density and dispersion profiles. One is then ensured from the start that the model is an equilibrium one (but stability may still be a question). Trial and error is then required to obtain useful models, with relevant density and velocity-dispersion profiles. Alternatively, we may start by dictating constraints on the more directly observable density, $\rho(r)$, and the runs of tangential and radial

velocity dispersions, $\sigma_t(r)$ and $\sigma_r(r)$, respectively. A necessary (but not sufficient) condition that there be an underlying stationary DF is that these three functions satisfy the Jeans equation

$$\frac{1}{\rho}\frac{d(\sigma_r^2\rho)}{dr}+\frac{2\beta\sigma_r^2}{r}=-\frac{M(r)G}{r^2}, \tag{1}$$

with

$$M(r)=4\pi\int_0^r r'^2\ \rho(r')\ dr', \tag{2}$$

and

$$\beta\equiv 1-\frac{\sigma_t^2}{\sigma_r^2}. \tag{3}$$

This is still only one functional constraint on the three functions, so to pinpoint a model we need two further functional constraint. For example, we can restrict ourselves to isotropic models and put $\sigma_r(r)=\sigma_t(r)=\sigma(r)$. Then we can start with a density profile and calculate $\sigma(r)$ from the Jeans equation, as in Tremaine et al. (1994). We can, alternatively, assume an equation of state–a relation between ρ and σ–and solve the Jeans equation for both $\rho(r)$ and $\sigma(r)$ (as for gas polytropes). Or, we may start with an assumed $\sigma(r)$ and solve the Jeans equation for $\rho(r)$ (as in the case of isothermal spheres).

If we want anisotropic models we may start with a different constraint on $\sigma_r(r),\ \sigma_t(r)$. For example, we may assume that their ratio is position independent (as we do in much of this paper), and then dictate either a density or a dispersion profile.

Such models are useful if at least some aspects of them can be given in simple, closed forms. In the end one has to ascertain that the model is underpinned by a positive DF satisfying the Jeans theorem, if one seeks to insure that the model is a realization of at least one equilibrium system.

We offer here what we believe is a novel family of such models. We start by dictating the run of velocity dispersions: a constant ratio between the tangential and radial dispersions, while each of them decreases, in a Keplerian manner, as $r^{-1/2}$. It is clear that in order to obtain a bound system with a finite total mass the dispersions must decline at least that fast asymptotically (lest they outstrip the escape velocity, which decreases in this manner). In fact, for a fixed-β the dispersions must behave so asymptotically. Such a behavior is also "natural" for a stellar system near a central, dominant point mass. We then extend this family of models by assuming that only $(\sigma_r^2+\sigma_t^2)^{1/2}$ is exactly Keplerian, but the anisotropy ratio is not constant.

The useful features of the models are: simple, closed-form expressions for the density, accumulated mass, velocity dispersions, and potential, and the effortless inclusion of a central point mass.

We describe the constant-β models in section 2, derive some additional properties in section 3 (including a partial discussion of underlying DFs), and explore a generalization in section 4.

2. THE MODELS

We substitute our assumed dispersion profile,

$$(1-\beta)^{-1}\sigma_t^2 = \sigma_r^2 = S/r, \tag{4}$$

in the Jeans equation(1) to obtain

$$\frac{dln\ \rho}{dln\ r} + 2\beta - 1 = -m(r), \tag{5}$$

where, $m(r) \equiv M(r)G/S$ is the dimensionless accumulated mass ($\hat{M} \equiv SG^{-1}$ has the dimensions of mass and can be used to normalize all the masses in the problem), and β is constant.

The addition of a point mass M_0 at the center is effected by subtracting from the right-hand side of eq.(5) the constant

$$m_0 \equiv M_0/\hat{M} = M_0G/S. \tag{6}$$

We then obtain for the Jeans equation

$$\frac{dln\ \rho}{dln\ r} + \gamma = -m(r), \tag{7}$$

where

$$\gamma \equiv 2\beta + m_0 - 1, \tag{8}$$

and $m(r)$ includes only the accumulated mass of the stellar "gas", excluding the point mass; we thus have $m(0) = 0$. Equation(7) can be solved analytically: We eliminate ρ using

$$\rho = \frac{\hat{M}}{4\pi r^2}m', \tag{9}$$

to get an equation for $m(r)$ (m' is the r-derivative of m)

$$rm'' + (\gamma-2)m' + mm' = 0. \tag{10}$$

The left-hand side is the derivative of $rm' + (\gamma-3)m + m^2/2$, which must then be a constant. Since $m(0) = 0$, this constant must be 0 so we get

$$m^{-1}(3-\gamma-\frac{1}{2}m)^{-1}m' = \frac{1}{r}. \tag{11}$$

For $\gamma \geq 3$, m' is negative, which is unphysical. For $\gamma < 3$, the equation is straightforwardly integrated to yield the desired mass profile

$$m(r) = 2(3-\gamma)\frac{x^{3-\gamma}}{1+x^{3-\gamma}}, \tag{12}$$

where $x = r/r_0$, and the scale radius, r_0, is an arbitrary integration constant; it equals the half-mass radius.

This mass profile corresponds to the density profile

$$\rho(r) = \frac{(3-\gamma)^2 \hat{M}}{2\pi r_0^3} \frac{x^{-\gamma}}{(1+x^{3-\gamma})^2}. \tag{13}$$

Near the origin, the density behaves as $r^{-\gamma}$, while at infinity it behaves as $r^{\gamma-6}$.

We see from eq.(12) that the total dimensionless mass is $m(\infty) = 2(3-\gamma)$, or

$$GM_g = 2(3-\gamma)S = 4(2-\beta)S - 2M_0 G. \tag{14}$$

S is thus determined by the two masses: the total "gas" mass, M_g, and the central point mass, M_0, through

$$S = [4(2-\beta)]^{-1} G(M_g + 2M_0). \tag{15}$$

This is the virial relation for the models. We shall see in section 3 that for $\gamma \geq 2$ the kinetic and potential energies of the model are infinite, but relation(15) still holds for all legitimate values of $\gamma < 3$, and follows directly from the structure equation.

The two masses determine the parameter m_0 through $m_0 = 4(2-\beta)M_0/(M_g + 2M_0)$, and γ through

$$\gamma = 2\beta - 1 + 4(2-\beta)\frac{M_0/M_g}{1 + 2M_0/M_g}. \tag{16}$$

In the limit of test-particle "gas", $M_g \to 0$, with β fixed, we have $\gamma \to 3$, so the model becomes meaningless. A meaningful test-particle model is obtained if we let $\beta \to -\infty$ (circular orbits) such that $\gamma < 3$; βM_g is then fixed at $(\gamma - 3)M_0/2$. Then $S \to 0$, but $2\sigma_t^2 r \to -2\beta S \to GM_0$, and we get a model with test particles on Keplerian circular orbits around a point mass, the likes of which can be built with any density distribution. If we admix test particles on non-circular orbits the Keplerian fall-off which we require cannot be maintained.

In the limit $\beta \to -\infty$, but M_g and M_0 kept constant, we have $\gamma \to -\infty$, and the sphere becomes an infinitely thin shell at an arbitrary radius r_0, with the particles moving on circular orbits: A self gravitating sphere cannot consist of particles on circular orbits with Keplerian velocities unless they are all at the same radius.

For $\gamma = 2$, one obtains the density distribution of the Jaffe model [Jaffe (1983)] [which is the same as the "eta" model of Tremaine et al. (1994), and Dehnen (1993) with $\eta = 1$]. This value of γ can be obtained only with a central mass as the maximum value of γ for $m_0 = 0$ is 1. In the isotropic case it corresponds to $M_0/M_g = 3/2$.

3. GENERAL PROPERTIES

Energies

We now consider the potential run of the model. The point mass makes the usual contribution to the potential; that of the stellar "gas", φ_g, can be straightforwardly integrated from the expression

$$\frac{d\varphi_g}{dr} = -\frac{GM(r)}{r^2} = -\frac{GM_g}{r_0^2}\frac{x^{1-\gamma}}{1+x^{3-\gamma}} \tag{17}$$

to give

$$\varphi_g(r) = -\frac{GM_g}{r}\ {}_2F_1[1,(3-\gamma)^{-1};(4-\gamma)(3-\gamma)^{-1};-(r_0/r)^{(3-\gamma)}], \tag{18}$$

where ${}_2F_1$ is the hypergeometric function [see e.g. Gradshteyn and Ryzhik (1980)]. As ${}_2F_1(a,b;c;0) = 1$ we have asymptotically $\varphi_g(r) \to -\frac{GM_g}{r}$, as expected.

As to the behavior of φ_g at the origin, we use the behavior of ${}_2F_1$ at large values of its argument to deduce that for $\gamma < 2$

$$\varphi_g(0) = -\frac{GM_g}{r_0}\Gamma\left(\frac{2-\gamma}{3-\gamma}\right)\Gamma\left(\frac{4-\gamma}{3-\gamma}\right). \tag{19}$$

For $\gamma = 2$, φ_g diverges logarithmically at the origin, and for $2 < \gamma < 3$, φ_g diverges as $r^{2-\gamma}$.

Some special cases: For $\gamma = 1$, one needs ${}_2F_1(1,1/2;3/2;-x^{-2}) = x\ tg^{-1}(1/x)$ to get

$$\varphi_g(r) = -\frac{GM_g}{r_0}tg^{-1}\left(\frac{r_0}{r}\right). \tag{20}$$

For $\gamma = 2$, ${}_2F_1(1,1;2;-x^{-1}) = x\ ln(1+1/x)$, yielding the known expression for the Jaffe model $\varphi_g(r) = (GM_g/r_0)ln[r/(r+r_0)]$.

The total kinetic energy of a model sphere is infinite for $\gamma \geq 2$, and for $\gamma < 2$ is

$$E_k = \frac{SM_g}{r_0}\frac{3-2\beta}{2}\Gamma\left(\frac{2-\gamma}{3-\gamma}\right)\Gamma\left(\frac{4-\gamma}{3-\gamma}\right). \tag{21}$$

The potential energy of the stellar "gas" is, for $\gamma < 2$,

$$E_p = -\frac{GM_g^2}{2r_0}\Gamma\left(\frac{2-\gamma}{3-\gamma}\right)\Gamma\left(\frac{4-\gamma}{3-\gamma}\right)\left(\frac{2-\gamma}{3-\gamma} + \frac{2M_0}{M_g}\right). \tag{22}$$

By eq.(15) they satisfy $E_k = -E_p/2$.

Projected properties

If we define the integrals

$$I(\mu,\nu;\lambda) \equiv \int_\lambda^\infty \frac{x^{-\mu}(x^2-\lambda^2)^{-1/2}}{(1+x^\nu)^2}dx, \tag{23}$$

then the projected surface density $\Sigma(a)$, at a projected radius $a = \lambda r_0$, is given by

$$\Sigma(a) = \frac{3-\gamma}{2\pi}\frac{M_g}{r_0^2}I(\gamma-1,3-\gamma;\lambda). \tag{24}$$

The line-of-sight (projected) velocity dispersion, $\sigma_\parallel(a)$, is

$$\sigma_\parallel^2(a) = \frac{S}{r_0}\frac{I(\gamma, 3-\gamma;\lambda) - \beta\lambda^2 I(\gamma+2, 3-\gamma;\lambda)}{I(\gamma-1, 3-\gamma;\lambda)}. \tag{25}$$

At large projected radii ($a \gg r_0$) we can write

$$\Sigma(a) \approx \frac{3-\gamma}{4\pi^{1/2}}\frac{M_g}{r_0^2}\frac{\Gamma\left(\frac{5-\gamma}{2}\right)}{\Gamma\left(\frac{6-\gamma}{2}\right)}\lambda^{\gamma-5}, \tag{26}$$

and the line-of-sight dispersion

$$\sigma_\parallel^2(a) \approx \frac{\Gamma^2\left(\frac{6-\gamma}{2}\right)}{\Gamma\left(\frac{7-\gamma}{2}\right)\Gamma\left(\frac{5-\gamma}{2}\right)}\left(1 - \beta\frac{6-\gamma}{7-\gamma}\right)\frac{S}{a}. \tag{27}$$

As examples, for $\gamma = 1$ we get asymptotically

$$\sigma_\parallel^2(a) \approx \frac{9\pi}{32}(1 - \frac{5}{6}\beta)\frac{S}{a}, \tag{28}$$

for $\gamma = 2$

$$\sigma_\parallel^2(a) \approx \frac{8}{3\pi}(1 - \frac{4}{5}\beta)\frac{S}{a}, \tag{29}$$

and for $\gamma \approx 3$

$$\sigma_\parallel^2(a) \approx \frac{\pi}{4}(1 - \frac{3}{4}\beta)\frac{S}{a}, \tag{30}$$

Near the origin ($a \ll r_0$) we can write, when $\gamma > 1$,

$$\Sigma(a) \approx \frac{3-\gamma}{4\pi^{1/2}}\frac{M_g}{r_0^2}\frac{\Gamma\left(\frac{\gamma-1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)}\lambda^{1-\gamma}, \tag{31}$$

and

$$\sigma_\parallel^2(a) \approx \frac{\Gamma^2\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(\frac{\gamma-1}{2}\right)}\left(1 - \beta\frac{\gamma}{\gamma+1}\right)\frac{S}{a}. \tag{32}$$

For example, for $\gamma = 2$

$$\sigma_\parallel^2(a) \approx \frac{2}{\pi}(1 - \frac{2}{3}\beta)\frac{S}{a}. \tag{33}$$

For $\gamma \approx 3$, the behaviors near the origin and at large radii are the same. We see that unlike the density, and surface-density, run, the projected-dispersion run depends on both β and γ. For $\gamma > 1$, $\sigma_\parallel$ declines as $r^{-1/2}$ at both ends; the proportionality constant is, in general, different, but may be the same for special choices of β and γ (for example, for $\gamma = 2$, $\beta = 5/6$).

Underlying distribution functions

We have not been able to ascertain that all the models discussed here have underlying non-negative DFs that satisfy the Jeans theorem. For the isotropic case, $\beta = 0$, one can try to

search for a DF that depends on the particle energy alone. The procedure is then straightforward. First, we note from eq.(4-139) in Binney and Tremaine (1987) that for this limited class of DFs the density must not increase with radius anywhere. This necessary condition excludes the (isotropic) models with $\gamma < 0$, for which ρ increases near the origin. The DF is uniquely determined by the density distribution through the Eddington relation [eq.(4-140b) of Binney and Tremaine (1987)]

$$f(\epsilon) = \frac{1}{8^{1/2}\pi^2}\left[\int_0^{\epsilon}\frac{d^2\rho}{d\Psi^2}\frac{d\Psi}{\sqrt{\epsilon-\Psi}} + \frac{1}{\sqrt{\epsilon}}\left(\frac{d\rho}{d\Psi}\right)_{\Psi=0}\right]. \tag{34}$$

Here, $\epsilon = -E$, where E is the energy, $\Psi = -\varphi$, and ρ is viewed as a function of Ψ as both are functions of r. The value $\Psi = 0$ occurs at radial infinity, where $\rho \propto r^{\gamma-6}$ decreases faster than $\Psi \propto r^{-1}$; thus, the second term in eq.(34) vanishes. If we change the integration variable to r, eq.(34) becomes

$$f(\epsilon) = \frac{1}{8^{1/2}\pi}\int_{\Psi^{-1}(\epsilon)}^{\infty} -\frac{d^2\rho}{d\Psi^2}\frac{d\Psi}{dr}\frac{dr}{\sqrt{\epsilon-\Psi}}. \tag{35}$$

One can show that

$$-\frac{d^2\rho}{d\Psi^2}\frac{d\Psi}{dr} = \rho\frac{m+\gamma}{m+\gamma+1}\left[\gamma - 1 + m\frac{m^2 + m(2\gamma+3/2) + (\gamma-1)(\gamma+3)}{(m+\gamma)(m+\gamma+1)}\right]. \tag{36}$$

Thus, for $\gamma \geq 1$ the integrand is non-negative, and the DF is always positive. For $\gamma < 1$, the integrand becomes negative for small enough radii, and the DF becomes negative for high enough energies. Such models do not have legitimate DFs that depend only on E.

The two limiting models discussed at the end of section 2 clearly also have legitimate underlying DFs.

4. A GENERALIZATION

Scott Tremaine (private communication) has pointed out to us that the crucial step in our derivation, of once integrating the Jeans equation [going from eq.(10) to eq.(11)], does not require that σ_r and σ_t are separately Keplerian, it is enough that $(\sigma_t^2 + \sigma_r^2)^{1/2}$ is. Making only this relaxed assumption permits us to explore a larger family of models, which we now proceed to do.

The general Jeans equation(1), in the presence of a central point mass M_0, can be brought to the form

$$(r^2 M'\sigma_r^2)' - [2r(\sigma_r^2 + \sigma_t^2) - GM_0]M' + \frac{G}{2}(M^2)' = 0, \tag{37}$$

where the derivation is with respect to r. Now assume only that

$$\sigma_r^2 + \sigma_t^2 = \hat{S}r^{-1}, \tag{38}$$

and, as before, define the dimensionless quantities $\hat{m}_0 \equiv GM_0/\hat{S}$, $\hat{m}(r) \equiv GM(r)/\hat{S}$, and also

$$s(r) \equiv \sigma_r^2(r)r/\hat{S} = \frac{1}{2-\beta(r)}. \tag{39}$$

Then, the Jeans equation becomes

$$(rs\hat{m}')' - \eta\hat{m}' + \frac{1}{2}(\hat{m}^2)' = 0, \tag{40}$$

where $\eta \equiv 2 - \hat{m}_0$. As explained in section 1, having imposed only one functional condition on the three functions ρ, σ_r, and σ_t, we have yet to specify another in order to pinpoint a model. This can be a dictation of the run of the anisotropy parameter, as we shall do below by dictating $s(r)$. Note that $0 \leq s(r) \leq 1$, where $s = 0, 1$ for purely tangential, and purely radial dispersion, respectively.

As before, the left-hand side of eq.(40) is a derivative, and is readily integrated once. As s is bounded, the integration constant vanishes, and we get

$$rs\hat{m}' - \eta\hat{m} + \frac{1}{2}\hat{m}^2 = 0. \tag{41}$$

Values of $\eta \leq 0$ are non-physical as they give a negative m'. For $\eta > 0$, the integration of eq.(41) gives the mass profile

$$\hat{m}(r) = \frac{2\eta e^{\eta X(r)}}{1+e^{\eta X(r)}}, \tag{42}$$

where

$$X(r) = \int_{r_0}^{r} \frac{dr}{rs(r)}, \tag{43}$$

and r_0 is some arbitrary radius that will appear as a scale in the density law. The integration constant in eq.(43) is chosen so that r_0 is the half-mass radius. Since s cannot exceed unity, $X(r)$ must diverge (at least logarithmically) for $r \to \infty$, and so $\hat{m}(\infty) = 2\eta$, yielding the virial relation

$$4\hat{S} = G(M_g + 2M_0). \tag{44}$$

Thus, η is determined by

$$\eta = \frac{2M_g}{M_g + 2M_0}. \tag{45}$$

The results for the constant-β case are reproduced when we note that in this case S and $\hat{S}$ are related by $\hat{S} = (2-\beta)S$, so $\hat{m} = m/(2-\beta)$, $\eta = (3-\gamma)/(2-\beta)$, etc..

The resulting density run is

$$\rho(r) = \frac{M_g}{2\pi r^3 s(r)} \frac{\eta e^{\eta X}}{(1+e^{\eta X})^2}. \tag{46}$$

At large r, where $X(r)$ is positive and diverges, the asymptotic form of the density is

$$\rho(r \to \infty) \approx \frac{\eta M_g}{2\pi r^3 s(r)} e^{-\eta X}; \tag{47}$$

near the origin, where $X(r)$ also diverges, but is negative,

$$\rho(r \to 0) \approx \frac{\eta M_g}{2\pi r^3 s(r)} e^{\eta X}. \tag{48}$$

One can try different forms of $s(r)$, subject to $0 \leq s(r) \leq 1$, and see if any interesting mass distributions result. Some general points first: If we depart from the constant-β models discussed above, a new radius scale must be introduced by the choice of s, beside the scale, r_0, introduced through the integration constant. This is because s must be of the form $s(r/a)$.

Near the origin the behavior is like that of the constant-β models with $\beta = \beta(0)$. Thus, if $s(0) \neq 0$ $[\beta(0) \neq -\infty]$, $X(r)$ diverges logarithmically (and is negative) at the origin, and we get from eqs.(43)(48) a power-law density behavior: $\rho \propto r^{-[3-\eta/s(0)]}$. If $s(0) = 0$, $X(r)$ diverges faster than logarithmically, and ρ vanishes non-analytically there [as in the constant $s(r) = 0$ case, where the "gas" is concentrated in an infinitely thin shell].

At large radii, if $s(\infty) \neq 0$, X diverges logarithmically, and ρ has a power-law behavior: $\rho \propto r^{-[3+\eta/s(\infty)]}$. If $s(\infty) = 0$, X diverges faster, and ρ vanishes faster than a power (e.g. as an exponential of a power).

Of the many models that can be produced we consider one class in more detail: Take $s(r)$ to vary monotonically between the values ν at $r = 0$, and τ at infinity as

$$s(r) = \frac{\nu + \tau(r/a)^d}{1 + (r/a)^d}. \tag{49}$$

Here, d is a power that controls the sharpness of the transition of s from ν to τ, and, when either vanishes, d measures the speed with which s does ($d \to -d$ is tantamount to $\nu \leftrightarrow \tau$). We take $\nu \neq 0$ ($\nu = 0$ can be treated by changing the sign of d), then, for $\tau \neq 0$, the integral in expression(43) for X gives

$$e^{\eta X} = \frac{x^\alpha(\nu + \tau x^d)^{\delta/d}}{x_0^\alpha(\nu + \tau x_0^d)^{\delta/d}}, \tag{50}$$

where $x \equiv r/a$, $x_0 \equiv r_0/a$, $\alpha \equiv \eta/\nu$, and $\delta \equiv \eta/\tau - \eta/\nu$. The mass profile is thus

$$\hat{m}(r) = \frac{2\eta x^\alpha(\nu + \tau x^d)^{\delta/d}}{x_0^\alpha(\nu + \tau x_0^d)^{\delta/d} + x^\alpha(\nu + \tau x^d)^{\delta/d}}, \tag{51}$$

and evinces the expected appearance of two distinct scale lengths. The density law is obtained by substituting expression(50) in eq.(46):

$$\rho(r) = \frac{\eta M_g}{2\pi r^3 s(r)} \frac{x^\alpha(\nu + \tau x^d)^{\delta/d} x_0^\alpha(\nu + \tau x_0^d)^{\delta/d}}{[x_0^\alpha(\nu + \tau x_0^d)^{\delta/d} + x^\alpha(\nu + \tau x^d)^{\delta/d}]^2}. \tag{52}$$

The density is a power law in radius at both ends with a power $-(3 - \eta/\nu)$ at the origin, and $-(3 + \eta/\tau)$ at infinity.

The case $\tau = 0$ requires special treatment: here one finds

$$e^{\eta X} = \left(\frac{x}{x_0}\right)^{\alpha} e^{(\alpha/d)(x^d - x_0^d)}, \tag{53}$$

giving rise to a density run that decreases exponentially at large r:

$$\rho(r) \propto r^{-(2+\alpha)} e^{-(\alpha/d)(r/a)^d}, \tag{54}$$

for $r \to \infty$.

We thus get a class of models that are more flexible as regards the density run, but they contain two scale lengths instead of one, and are certainly less wieldy.

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